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Open Wilson lines as closed strings in non-commutative field theories*

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Abstract. Open Wilson line operators and the generalized star product have been studied extensively in non-commutative gauge theories. We show that they also show up in non-commutative scalar field theories as universal structures. We first point out that the dipole picture of non-commutative geometry provides an intuitive argument for robustness of the open Wilson lines and generalized star products therein. We calculate the one-loop effective action of the non-commutative scalar field theory with cubic self-interaction and show explicitly that the generalized star products arise in the non-planar part. It is shown that, in the low-energy, large non-commutativity limit, the non-planar part is expressible solely in terms of the *scalar* open Wilson line operator and descendants, the latter being interpreted as composite operators representing a closed string.

1 Introduction and summary

One of the most salient features of non-commutative field theories is that physical excitations are described by *"dipoles"* – weakly interacting, non-local objects. Let us denote their center-of-mass momentum and dipole moment $\boldsymbol{k}$ and $\Delta\boldsymbol{x}$, respectively. According to the "dipole" picture, originally developed in [1] and more recently reiterated in [2], the two are related:

$$\Delta\boldsymbol{x}^a = \theta^{ab}\boldsymbol{k}_b. \tag{1}$$

Here, θ^{ab} denotes the non-commutativity parameter:

$$\left\{\boldsymbol{x}^a, \boldsymbol{x}^b\right\}_\star = \mathrm{i}\theta^{ab}, \tag{2}$$

in which $\{\cdot\}$ refers to the Moyal commutator, defined in terms of the $\star$-product:

$$\{A(x_1)B(x_2)\}_\star := \exp\left(\frac{\mathrm{i}}{2}\partial_1\wedge\partial_2\right)A(x_1)B(x_2), \tag{3}$$

where

$$\partial_1\wedge\partial_2 := \theta^{ab}\partial_1^a\partial_2^b.$$

Evidently, in the commutative limit, the dipoles shrink in size and represent pointlike excitations.

In non-commutative *gauge* theories, (part of) the gauge orbit is the same as the translation along the non-commutative directions [3–5]. For example, in non-commutative U(1) gauge theory, the gauge potential $\boldsymbol{A}_\mu(x)$ and the neutral scalar field $\Phi(x)$, both of which give rise to "dipoles", transform in the "adjoint" representation:

$$\begin{aligned}\delta_\epsilon\boldsymbol{A}_\mu(x) &= \mathrm{i}\int\frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\widetilde{\epsilon}(\boldsymbol{k})\\ &\quad\times\left[(\boldsymbol{A}_\mu(\boldsymbol{x}+\theta\cdot\boldsymbol{k})-\boldsymbol{A}_\mu(\boldsymbol{x}-\theta\cdot\boldsymbol{k}))+\mathrm{i}\boldsymbol{k}_\mu\right]\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}},\\ \delta_\epsilon\Phi(x) &= \mathrm{i}\int\frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\widetilde{\epsilon}(\boldsymbol{k})\\ &\quad\times\left[\Phi(\boldsymbol{x}+\theta\cdot\boldsymbol{k})-\Phi(\boldsymbol{x}-\theta\cdot\boldsymbol{k})\right]\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}},\end{aligned} \tag{4}$$

where the infinitesimal gauge transformation parameter is denoted

$$\epsilon(\boldsymbol{x}) = \int\frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\widetilde{\epsilon}(\boldsymbol{k}).$$

For fields transforming in "adjoint" representations under the non-commutative gauge group, it has been shown that the only physical observables are the "open Wilson lines" [3–6] along an open contour C:

$$\begin{aligned}W_{\boldsymbol{k}}[C] &= \mathcal{P}_\mathrm{t}\int\mathrm{d}^2\boldsymbol{x}\\ &\quad\times\exp_\star\left[\mathrm{i}\int_0^1\mathrm{d}t\dot{\boldsymbol{y}}(t)\cdot\boldsymbol{A}(x+\boldsymbol{y}(t))\right]\star\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}},\end{aligned} \tag{5}$$

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and their descendant operators. The $\star$-product refers to the base point $\boldsymbol{x}$ of the open contour C. Despite being defined over an open contour, the operator is gauge invariant *provided* the momentum $\boldsymbol{k}$ is related to the geodesic distance $\boldsymbol{y}(1) - \boldsymbol{y}(0) := \Delta\boldsymbol{x}$ precisely by the "dipole relation", (1). In other words, in non-commutative gauge theory, the open Wilson lines (physical observables) ought to obey the dipole relation (1) as an immediate consequence of the gauge invariance! For a straight Wilson line, expanding (5) in successive powers of the gauge field $\boldsymbol{A}_\mu$, it was observed [7,8] that the generalized $\star$-product, $\star_n$, a structure discovered first in [9,10], emerge:

$$W_{\boldsymbol{k}}[C] = \int \mathrm{d}^2\boldsymbol{x} \times \left[1 - (\partial \wedge \boldsymbol{A}) + \frac{1}{2!}(\partial \wedge \boldsymbol{A})^2_{\star_2} + \cdots\right] \star \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}. \quad (6)$$

Apparently, the generalized $\star_n$ products give rise to different algebraic structures from Moyal's $\star$-product. For instance, the first two, $\star_2, \star_3$, defined by

$$[A(x_1)B(x_2)]_{\star_2} := \frac{\sin\left(\frac{1}{2}\partial_1 \wedge \partial_2\right)}{\frac{1}{2}\partial_1 \wedge \partial_2} A(x_1)B(x_2),$$

$$[A(x_1)B(x_2)C(x_3)]_{\star_3} := \left[\frac{\sin\left(\frac{1}{2}\partial_2 \wedge \partial_3\right)}{\frac{1}{2}(\partial_1 + \partial_2) \wedge \partial_3} \frac{\sin\left(\frac{1}{2}\partial_1 \wedge (\partial_2 + \partial_3)\right)}{\frac{1}{2}\partial_1 \wedge (\partial_2 + \partial_3)} + (1 \leftrightarrow 2)\right] \times A(x_1)B(x_2)C(x_3),$$

show that the $\star_n$'s are commutative but non-associative. Even though the definition of the open Wilson line is given in terms of the path-ordered $\star$-product, its expansion in powers of the gauge potential involves the generalized $\star_n$-product at each nth order. The complicated $\star_n$ products have arisen upon expansion in powers of the gauge potential, and are attributable again to the dipole nature of the Wilson line and the gauge invariance thereof – each term in (6) is *not* gauge invariant, as the gauge transformation (4) mixes terms involving different $\star_n$'s. Indeed, the generalized $\star_n$ products are not arbitrary but obey recursive identities:

$$\mathrm{i}\left[\partial_x A(x) \wedge \partial_x B(x)\right]_{\star_2} = \{A(x), B(x)\}_\star,$$
$$\mathrm{i}\partial_x \wedge \left[A(x)B(x)\partial_x C(x)\right]_{\star_3} = A(x) \star_2 \{B(x), C(x)\}_\star + B(x) \star_2 \{A(x), C(x)\}_\star .$$

These recursive identities are crucial for ensuring gauge invariance of the power-series expanded open Wilson line operator, (6).

The open Wilson lines and the generalized $\star$-products therein have been studied extensively in the literature, largely in the context of non-commutative *gauge* theories. On the other hand, the dipole relation (1) ought to be a universal relation, applicable for *any* theories defined over non-commuative spacetime. The aforementioned remark that non-commutative gauge invariance plays a prominent role in both deriving the relation (1) and the recursive relation among the generalized $\star$-products then raises an immediate question. Are these structures present also in non-commutative field theories, in which neither gauge invariance nor gauge field exists?

In this paper, we show that the answer to the above question is affirmatively *positive*, by studying d-dimensional massive $\lambda\left[\Phi^3\right]_\star$ theory. We compute the one-loop effective action and find that non-planar contributions are expressible in terms of the generalized $\star$-product. Most significantly, in the low-energy and large non-commutativity limit, we show that the non-planar part of the one-loop effective action can be resummed into a remarkably simple expression:

$$\Gamma_{\mathrm{np}}[\Phi] = \frac{\hbar}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} W_{\boldsymbol{k}}[\Phi]\widetilde{\mathcal{K}_{-d/2}}\left(\boldsymbol{k} \circ \boldsymbol{k}\right) W_{-\boldsymbol{k}}[\Phi], \quad (7)$$

where $W_{\boldsymbol{k}}[\Phi]$ denotes the first descent of the scalar Wilson line operators:

$$W_{\boldsymbol{k}}[\Phi] := \mathcal{P}_\mathrm{t} \int \mathrm{d}^2\boldsymbol{x} \times \exp\left(-g\int_0^1 \mathrm{d}t|\dot{\boldsymbol{y}}(t)|\Phi(x + \boldsymbol{y}(t))\right) \star \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \quad (8)$$

$$(\Phi^n W)_{\boldsymbol{k}}[\Phi] := \left(-\frac{\partial}{\partial g}\right)^n W_{\boldsymbol{k}}[\Phi] \quad \left(g := \frac{\lambda}{4m}\right). \quad (9)$$

Here, $\widetilde{\mathcal{K}_{-d/2}}$ denotes a (Fourier transformed) kernel and $\boldsymbol{k} \circ \boldsymbol{k} := (\theta \cdot \boldsymbol{k})^2 \equiv \ell^2$, etc. We trust that the above result bears considerable implications to the non-commutative solitons studied in [11,12] and to the UV/IR mixing discovered in [13], and we will report the details elsewhere.

2 Scalar open Wilson lines as dipoles

2.1 Dipole relation for scalar fields

We begin with a proof that the dipole relation (1) holds for scalar fields as well, wherein no gauge invariance is present. Recall that, in non-commutative spacetime, energy-momentum is a good quantum number, as the non-commutative geometry is invariant under translation along the non-commutative directions. As such, consider the following set of operators, the so-called Parisi operators [14]:

$$\mathcal{O}_n(x_1, \cdots, x_n; \boldsymbol{k}) = \int \mathrm{d}^2\boldsymbol{z} \times \Phi_1(x_1 + \boldsymbol{z}) \star \Phi_2(x_2 + \boldsymbol{z}) \star \cdots \star \Phi_n(x_n + \boldsymbol{z}) \star \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}},$$

viz. the Fourier transform of a string of elementary fields, $\Phi_k(x)$ $(k = 1, 2, \cdots)$. Consider the one-point function:

$$G_1(x, \boldsymbol{k}) = \langle \mathcal{O}_2(x, \boldsymbol{k})\rangle = \left\langle \int \mathrm{d}^2\boldsymbol{z}\Phi(\boldsymbol{z}) \star \Phi(x + \boldsymbol{z}) \star \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \right\rangle .$$

In terms of Fourier decomposition of the scalar field,

$$\Phi(x) = \int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \widetilde{\Phi}(\boldsymbol{k}),$$

one obtains

$$G_1(x, \boldsymbol{k}) = \int \frac{\mathrm{d}^2\boldsymbol{l}}{(2\pi)^2} \widetilde{\Phi}(\boldsymbol{l}) \widetilde{\Phi}(-\boldsymbol{l}+\boldsymbol{k}) \times \exp\left[\mathrm{i}\boldsymbol{l}\cdot\left(x + \frac{1}{2}\theta\cdot\boldsymbol{k}\right)\right].$$

Consider the "wave-packet" of the scalar particle, $\Phi(\boldsymbol{z}) = \Phi_0 \delta^{(2)}(\boldsymbol{z})$ and $\Phi(x+\boldsymbol{z}) = \Phi_0 \delta^{(2)}(x+\boldsymbol{z})$ so that $\widetilde{\Phi}(\boldsymbol{l}) = \Phi_0 \exp(\mathrm{i}\boldsymbol{l}\cdot x)$. From the above equation, one then finds that

$$G_1(x, \boldsymbol{k}) = \Phi_0^2 \delta^{(2)}\left(x + \frac{1}{2}\theta\cdot\boldsymbol{k}\right).$$

Thus, one finds that the stationary point of the correlator is given by

$$\Delta x^a \sim \theta^{ab}\boldsymbol{k}_b, \tag{10}$$

and hence precisely by the "dipole relation", (1).

2.2 Scalar open Wilson lines and generalized star products

First, as in the case of the *gauge* open Wilson lines, (5), we will show that power-series expansion of the *scalar* open Wilson line operators (9) gives rise to generalized $\star_n$ products, and take this as the definition of the products for "adjoint" scalar fields.

We begin with a remark concerning the symmetry of the open Wilson line operators. The open Wilson line operator (5) possesses reparameterization invariance:

$$t \to \tau(t) \tag{11}$$

such that $\tau(0) = 0$, $\tau(1) = 1$ and $\dot{\tau}(t) > 0$, as the gauge element $\mathrm{d}\boldsymbol{y}(t)\cdot\boldsymbol{A}(x+\boldsymbol{y})$ is invariant under the reparameterization, (11):

$$\begin{aligned} &\mathrm{d}t\dot{\boldsymbol{y}}(t)\cdot\boldsymbol{A}(x+\boldsymbol{y}(t)) \\ &\quad = \left(\mathrm{d}\tau|\dot{\tau}(t)|^{-1}\right)\left(\dot{\tau}(t)\dot{\boldsymbol{y}}(\tau)\right)\cdot\boldsymbol{A}(x+\boldsymbol{y}(\tau)) \\ &\quad = \mathrm{d}\tau\dot{\boldsymbol{y}}(\tau)\cdot\boldsymbol{A}(x+\boldsymbol{y}(\tau)). \end{aligned}$$

Likewise, the *scalar* open Wilson line operator (9) possesses reparameterization invariance, as the scalar element $|\mathrm{d}\boldsymbol{y}(t)|\Phi(x+\boldsymbol{y})$ is invariant under the reparameterization, (11):

$$\begin{aligned} &\mathrm{d}t|\dot{\boldsymbol{y}}(t)|\Phi(x+\boldsymbol{y}(t)) \\ &\quad = \left(\mathrm{d}\tau|\dot{\tau}(t)|^{-1}\right)\left(|\dot{\tau}(t)\dot{\boldsymbol{y}}(\tau)|\right)\Phi(x+\boldsymbol{y}(\tau)) \\ &\quad = \mathrm{d}\tau|\dot{\boldsymbol{y}}(\tau)|\Phi(x+\boldsymbol{y}(\tau)). \end{aligned}$$

In fact, the reparameterization invariance, along with rotational invariance on the non-commutative subspace and reduced Lorentz invariance on the complement spacetime, puts a constraint powerful enough to determine uniquely the structure of the scalar open Wilson line operator, (9), much as for that of the gauge open Wilson line operator, (5).

For illustration, we will examine a simplified form of the scalar open Wilson line with insertion of a local operator $\mathcal{O}$ at the location $\boldsymbol{R}$ on the Wilson line:

$$(\mathcal{O}_{\boldsymbol{R}}W)_k[\Phi] := \mathcal{P}_\mathrm{t}\int \mathrm{d}^2\boldsymbol{x}\mathcal{O}(x+\boldsymbol{R}) \star\exp\left(-g\int_0^1 \mathrm{d}t|\dot{\boldsymbol{y}}(t)|\Phi(x+\boldsymbol{y}(t))\right)\star \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}.$$

Take a *straight* Wilson line:

$$\boldsymbol{y}(t) = \boldsymbol{L}t, \quad \text{where} \quad \boldsymbol{L}^a = \theta^{ab}\boldsymbol{k}_b := (\theta\cdot\boldsymbol{k})^a, \quad L := |\boldsymbol{L}|,$$

corresponding to a uniform distribution of the momentum $\boldsymbol{k}$ along the Wilson line. As the path-ordering progresses to the right with increasing t, the power-series expansion in $gL\Phi$ yields

$$\begin{aligned} (\mathcal{O}_{\boldsymbol{R}}W)_\mathrm{k}[\Phi] &= \int \mathrm{d}^2\boldsymbol{x}\mathcal{O}(\boldsymbol{x}+\boldsymbol{R})\star \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \\ &+(-gL)\int \mathrm{d}^2\boldsymbol{x}\int_0^1 \mathrm{d}t\mathcal{O}(x+\boldsymbol{R})\star\Phi(x+\boldsymbol{L}t)\star \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \\ &+(-gL)^2\int \mathrm{d}^2\boldsymbol{x}\int_0^1 \mathrm{d}t_1\int_{t_1}^1 \mathrm{d}t_2 \\ &\times\mathcal{O}(\boldsymbol{x}+\boldsymbol{R})\star\Phi(x+\boldsymbol{L}t_1)\star\Phi(x+\boldsymbol{L}t_2)\star \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} + \cdots \end{aligned}$$

Each term can be evaluated, for instance, by Fourier transforming $\mathcal{O}$ and the Φ's:

$$\mathcal{O}(x) = \int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\widetilde{O}(\boldsymbol{k})T_{\boldsymbol{k}}, \quad \Phi(x) = \int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\widetilde{\Phi}(\boldsymbol{k})T_{\boldsymbol{k}},$$

where

$$T_{\boldsymbol{k}} = \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}},$$

taking the $\star$-products of the translation generators $T_{\boldsymbol{k}}$'s

$$T_{\boldsymbol{k}}\star T_{\boldsymbol{l}} = \mathrm{e}^{(\mathrm{i}/2)\boldsymbol{k}\wedge\boldsymbol{l}}T_{\boldsymbol{k}+\boldsymbol{l}},$$

and then evaluating the parametric $t_1, t_2, \cdots$ integrals. Fourier transforming back to the configuration space, after a straightforward calculation, one obtains

$$\begin{aligned} (\mathcal{O}_{\boldsymbol{R}}W)_\mathrm{k}[\Phi] &= \int \mathrm{d}^2\boldsymbol{x}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathcal{O}(x+\boldsymbol{R}) \\ &+(-gL)\int \mathrm{d}^2\boldsymbol{x}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\left[\mathcal{O}(x+\boldsymbol{R})\Phi(x)\right]_{\star_2} \\ &+\frac{1}{2!}(-gL)^2\int \mathrm{d}^2\boldsymbol{x}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\left[\mathcal{O}(x+\boldsymbol{R})\Phi(x)\Phi(x)\right]_{\star_3} \\ &+\cdots, \end{aligned} \tag{12}$$

where the products involved, $\star_2, \star_3, \cdots$, are precisely the same generalized star products as those appearing in the *gauge* open Wilson line operators.

In fact, for the straight open Wilson lines, one can take each term of the Taylor expansion in (12) as the definition of the generalized $\star_n$ products. Denote the exponent of the scalar open Wilson line operator as

$$\Phi_\ell(x,\boldsymbol{k}) := \int_0^1 \mathrm{d}t_\ell |\theta\cdot\boldsymbol{k}| \Phi(x+\theta\cdot\boldsymbol{k}t_\ell).$$

We have taken the open Wilson lines as straight ones. Then, analogous to [10], the generalized $\star_n$ products are given by

$$\begin{aligned}[\Phi\star_n\Phi]_{\boldsymbol{k}} &:= \mathcal{P}_{\mathrm{t}}\int \mathrm{d}^d x \\ &\quad\times\Phi_1(x,\boldsymbol{k})\star\Phi_2(x,\boldsymbol{k})\cdots\Phi_n(x,\boldsymbol{k})\star \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \\ &= \int \mathrm{d}^d x \Phi^{\star n}(x)\star \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}. \qquad (13)\end{aligned}$$

3 One-loop effective action in $\lambda[\Phi^3]_\star$ theory

Given the above results, we assert that the *scalar* open Wilson line operators play a central role in non-commutative field theories. To support our assertion, we now show that, for the case of $\lambda[\Phi^3]_\star$ scalar field theory, the effective action is expressible entirely in terms of the open Wilson line operators and nothing else. Begin with the classical action

$$S_{\mathrm{NC}} = \int \mathrm{d}^d x\left[\frac{1}{2}(\partial_\mu\Phi)^2 - \frac{1}{2}m^2\Phi^2 - \frac{\lambda}{3!}\Phi\star\Phi\star\Phi\right]. \qquad (14)$$

At one-loop order, the effective action can be obtained via the background field method[1]: expand the scalar field around a classical configuration, $\Phi = \Phi_0 + \varphi$,

$$S_{\mathrm{NC}} = \int \mathrm{d}^d x \frac{1}{2}\varphi(x)\star\left[-\partial_x^2 - m^2 - \lambda\Phi_0(x)\right]\star\varphi(x), \qquad (15)$$

and then integrate out the quantum flucutation, φ. Fourier transforming yields the interaction vertex of the form

$$\begin{aligned}\mathrm{i}S_{\mathrm{int}} &= -\mathrm{i}\frac{\lambda}{2\cdot 2}\int\prod_{a=1}^{3}\frac{\mathrm{d}^d k_a}{(2\pi)^d}\widetilde{\varphi}(k_1)\widetilde{\Phi_0}(k_2)\widetilde{\varphi}(k_3) \\ &\quad\times\left[\mathrm{e}^{-(\mathrm{i}/2)k_1\wedge k_2} + \mathrm{e}^{+(\mathrm{i}/2)k_1\wedge k_2}\right] \\ &\quad\times(2\pi)^d\delta^{(d)}(k_1+k_2+k_3) \\ &:= (\boldsymbol{U}+\boldsymbol{T}), \qquad (16)\end{aligned}$$

under the rule that the φ's are Wick contracted in the order they appear in the Taylor expansion of $\exp(\mathrm{i}S_{\mathrm{int}})$. The $\boldsymbol{U}$ and $\boldsymbol{T}$ parts refer to the untwisted and the twisted vertex interactions, respectively. In (16), the relative sign between $\boldsymbol{T}$ and $\boldsymbol{U}$ is +, opposite to those for the gauge fields. This can be traced to the fact that, under "time-reversal" $t\to(1-t)$, the exponents in gauge and scalar open Wilson lines, (5) and (9), transform as even and odd, respectively. The N-point correlation function is given by $(\mathrm{i}S_{\mathrm{int}})^N/N!$ term in the Taylor expansion and, as shown by Filk [16], consists of planar and non-planar contributions. In the binomial expansion of $(\mathrm{i}S_{\mathrm{int}})^N = (\boldsymbol{U}+\boldsymbol{T})^N$, the two terms, $[\boldsymbol{U}]^N$ and $[\boldsymbol{T}]^N$, yield the planar contribution, while the rest comprises the non-planar contribution. Evaluation of the N-point correlation function is straightforward. Summing over N, the one-loop effective action is given by

$$\begin{aligned}\Gamma[\Phi_0] &:= -\frac{\hbar}{2}\ln\det\left[-\partial_x^2 - m^2 - \lambda\Phi_0(x)\star - \mathrm{i}\epsilon\right]_\star \\ &= \sum_{\mathrm{N}=1}^{\infty}\int\prod_{\ell=1}^{N}\frac{\mathrm{d}^d q_\ell}{(2\pi)^d}\widetilde{\Phi_0}(q_1)\cdots\widetilde{\Phi_0}(q_N) \\ &\quad\times\ \boldsymbol{\Gamma}^{(\mathrm{N})}[q_1,\cdots,q_N].\end{aligned}$$

Here, the N-point correlation function takes schematically the following form:

$$\begin{aligned}\boldsymbol{\Gamma}^{(\mathrm{N})}[q_1,\cdots,q_N] &= (2\pi)^d\delta^{(d)}(q_1+\cdots+q_N) \qquad (17) \\ &\quad\times\mathcal{S}_N\cdot\mathcal{A}(q_1,\cdots,q_N)\cdot\mathcal{B}(q_1,\cdots,q_N),\end{aligned}$$

where $\mathcal{S}_N$ refers to the combinatorics factor, $\mathcal{A}$ is the Feynman loop integral, and $\mathcal{B}$ consists of the non-commutative phase factors.

3.1 Effective action: Non-planar part

We are primarily interested in the non-planar part. After some straightforward algebra, we have obtained the non-planar part as a double sum involving the generalized star products:

$$\begin{aligned}\Gamma_{\mathrm{np}} &= \hbar\sum_{\mathrm{N}=2}^{\infty}\left(-\frac{\lambda}{4}\right)^N\frac{1}{N!}\int\mathrm{d}^d x \\ &\quad\times\left(\frac{1}{2}\sum_{n=1}^{N-1}\binom{N}{n}[\Phi_0\star_n\Phi_0](x)\right. \\ &\quad\left.\times\ \mathcal{K}_{N-(d/2)}\cdot[\Phi_0\star_{N-n}\Phi_0](x)\right). \qquad (18)\end{aligned}$$

Here, $[\Phi_0\star_n\Phi_0](x)$ refers to the generalized $\star_n$ products of the Φ_0's and the combinatorial factor

$$C_{\{N\}} = \frac{1}{2}\binom{N}{n}$$

originates from the binomial expansion of $(\boldsymbol{U}+\boldsymbol{T})^N$ modulo an inversion symmetry, corresponding to Hermitian conjugation. The kernel $\mathcal{K}_n$ is defined by

$$\mathcal{K}_n := \mathcal{K}_n(-\partial_x\circ\partial_x), \qquad (19)$$

where

$$\mathcal{K}_n(z^2) = 2\left(\frac{1}{2\pi}\right)^{d/2}\left(\frac{|z|}{m}\right)^n \boldsymbol{K}_n(m|z|)$$

[1] Technical details of the following results will be reported in a separate paper [15]

in terms of the modified Bessel functions, $\boldsymbol{K}_n$.

To proceed, we will be taking the low-energy, large non-commutativity limit:

$$q_\ell \sim \epsilon, \quad \mathrm{Pf}\theta \sim \frac{1}{\epsilon^2} \quad \text{as} \quad \epsilon \to 0, \tag{20}$$

so that

$$q_\ell \cdot q_m \sim \mathcal{O}(\epsilon^{+2}) \to 0, \quad q_\ell \wedge q_m \to \mathcal{O}(1),$$
$$q_\ell \circ q_m \sim \mathcal{O}(\epsilon^{-2}) \to \infty. \tag{21}$$

In this limit, the modified Bessel function $\boldsymbol{K}_n$ exhibits the following asymptotic behavior:

$$\boldsymbol{K}_n(mz) \to \sqrt{\frac{\pi}{2mz}}\mathrm{e}^{-m|z|}\left[1 + \mathcal{O}\left(\frac{1}{m|z|}\right)\right]. \tag{22}$$

Most importantly, the asymptotic behavior is *independent* of the index n. Hence, in the low-energy limit, the Fourier transformed kernels, $\widetilde{\mathcal{K}_n}$, obey the following recursive relation:

$$\widetilde{\mathcal{K}}_{n+1}\left(\boldsymbol{k} \circ \boldsymbol{k}\right) = \left(\frac{|\theta \cdot \boldsymbol{k}|}{m}\right)\widetilde{\mathcal{K}}_n\left(\boldsymbol{k} \circ \boldsymbol{k}\right), \tag{23}$$

viz.

$$\widetilde{\mathcal{K}_n}\left(\boldsymbol{k} \circ \boldsymbol{k}\right) = \left(\frac{|\theta \cdot \boldsymbol{k}|}{m}\right)^n \widetilde{\mathcal{Q}}\left(\boldsymbol{k} \circ \boldsymbol{k}\right). \tag{24}$$

Here, the kernel $\widetilde{\mathcal{Q}}$ is given by

$$\widetilde{\mathcal{Q}}\left(\boldsymbol{k} \circ \boldsymbol{k}\right) = (2\pi)^{(1-d)/2}\left|\frac{1}{m\theta \cdot \boldsymbol{k}}\right|^{1/2}\exp(-m|\theta \cdot \boldsymbol{k}|). \tag{25}$$

Note that in the power-series expansion of the effective action the natural expansion parameter is $|\theta \cdot \boldsymbol{k}|^2$.

Thus, the non-planar one-loop effective action in momentum space is expressible as

$$\begin{aligned}\Gamma_{\mathrm{np}}[\Phi] = {}& \frac{\hbar}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\widetilde{\mathcal{K}}_{-d/2}\left(\boldsymbol{k} \circ \boldsymbol{k}\right)\sum_{N=2}^{\infty}\sum_{n=1}^{N-1}\left(-\frac{\lambda}{4m}\right)^N \\ & \times \left(\frac{1}{n!}|\theta \cdot \boldsymbol{k}|^n\left[\widetilde{\Phi} \star_n \widetilde{\Phi}\right]_{\boldsymbol{k}}\right) \\ & \times \left(\frac{1}{(N-n)!}|\theta \cdot \boldsymbol{k}|^{N-n}\left[\widetilde{\Phi} \star_{N-n} \widetilde{\Phi}\right]_{-\boldsymbol{k}}\right).\end{aligned} \tag{26}$$

Utilizing the relation between the generalized $\star_n$ products and the *scalar* open Wilson line operators, as elaborated in Sect. 2, the non-planar one-loop effective action can be summed up in a remarkably simple closed form. Denote the rescaled coupling parameter by $g := \lambda/4m$ (see (9)). Exploiting the exchange symmetry $n \leftrightarrow (N-n)$, one can rearrange the summations into *decoupled* ones over n and $(N-n)$. Moreover, because of $[\widetilde{\Phi} \star_0 \widetilde{\Phi}]_{\boldsymbol{k}} = (2\pi)^d \delta^{(d)}(k)$, the summations can be extended to $n = 0$, $(N-n) = 0$ terms, as they yield identically vanishing contribution after k-integration[3]. One finally obtains

$$\begin{aligned}\Gamma_{\mathrm{np}}[\Phi] = {}& \frac{\hbar}{2}\int \frac{\mathrm{d}^d k_1}{(2\pi)^d}\frac{\mathrm{d}^d k_2}{(2\pi)^d}(2\pi)^d \delta^d(k_1 + k_2) \\ & \times W_{\boldsymbol{k}_1}[\Phi] \cdot \widetilde{\mathcal{K}_{-d/2}}\left(-\boldsymbol{k}_1 \circ \boldsymbol{k}_2\right) \cdot W_{\boldsymbol{k}_2}[\Phi],\end{aligned}$$

yielding precisely the aforementioned result, (7).

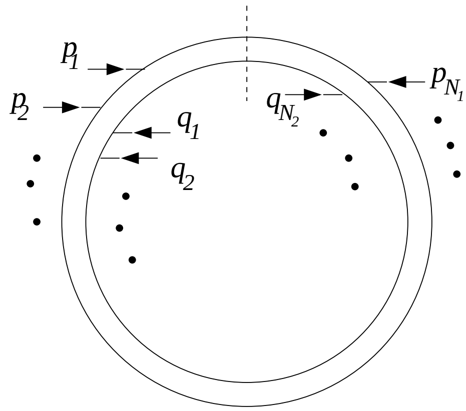

Fig. 1. Feynman diagram of one-loop, non-planar N-point correlation function. The simplest situation is when the Nth interaction is twisted, as discussed in the text, viz. $q_1 = q_1, \cdots, q_{N_1} = q_{N-1}$, and $p_1 = q_N$, $p_2 = p_3 = \cdots = 0$

3.2 Explicit calculations

To convince the readers that the expression (7) is indeed correct, we will evaluate below the simplest yet non-trivial correlation functions: $N = 3, 4$. Utilizing the factorization property [21], begin with non-planar N-point correlation functions in which one of the external legs is twisted. Denoting the twisted vertex as the Nth, the relevant Feynman diagram is given in Fig. 1. Evaluating Fig. 1 explicitly, one obtains the non-planar one-loop correlation functions of the form (17). The Feynman loop integral $\mathcal{A}$ is independent of the non-commutativity and hence has the same form as in the commutative counterpart. In the parameterization of the internal and the external momenta as in Fig. 1, after Wick rotation to Euclidean spacetime, the loop integral is given by

$$\begin{aligned}&\mathcal{A}\left(q_1, \cdots, q_N\right) \\ &\quad = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\prod_{\ell=0}^{N-1}\frac{1}{(k + q_1 + \cdots + q_\ell)^2 + m^2}.\end{aligned} \tag{27}$$

Introduce the Schwinger–Feynman parameterization for each propagator. This leads to

$$\mathcal{A} = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\int_0^\infty \prod_{\ell=1}^{N}\mathrm{d}\alpha_\ell \exp\left[-\alpha(m^2 + k^2)\right.$$

[2] As will be shown in a moment, this implies that the manifestly reparameterization invariant open Wilson line operator (9) is more fundamental than those defined in [17] and utilized in [18–20]

[3] We assume that the non-commutativity is turned on only on two-dimensional subspace

$$- 2\left(\sum_{m=1}^{N-1}\alpha_{m+1}(q_1+\cdots+q_m)\right)\cdot k + \mathcal{O}(q^2)\Bigg],$$

where $\alpha := \sum_{\ell=1}^{N}\alpha_\ell$ serves as the moduli measuring perimeter of the one-loop graph. As we are mainly concerned with the low-energy, large non-commutativity limit, (20) and (21), we will drop terms of order $\mathcal{O}(q^2)$ in what follows.

The Moyal phase factor, $\mathcal{B}$, is extractible from the $\boldsymbol{U}$ and $\boldsymbol{T}$ factors for untwisted and twisted interaction vertices, respectively. One finds, in counter-clockwise convention for the Wick contraction,

$$\mathcal{B}(q_1,\cdots,q_N) = \exp\left[\mathrm{i}k\wedge\left(\sum_{\ell=1}^{N-1}q_\ell\right)\right]\exp\left[+\frac{\mathrm{i}}{2}\sum_{\ell<m}^{N-1}q_\ell\wedge q_m\right].$$

The integrand in $\mathcal{A}$ is simplified once the loop momentum variable is shifted:

$$k^\mu \longrightarrow k^\mu - \frac{1}{\alpha}\left[\sum_{m=1}^{N-1}\alpha_{m+1}(q_1+\cdots+q_m)\right].$$

Change the Schwinger–Feynman moduli parameters into those for the ordered interaction vertices around the circumference of the one-loop diagram:

$$\alpha_\ell \longrightarrow \alpha(x_{\ell-1}-x_\ell),$$

where

$$(1 > x_1 > x_2 > \cdots > x_{N-1} > 0).$$

Accordingly, $\mathcal{B}$ is multiplied by a moduli-dependent phase factor,

$$\mathcal{B}(q_1,\cdots,q_N) \longrightarrow \mathcal{B}(q_1,\cdots,q_N)\times\exp\left(-\mathrm{i}\sum_{\ell<m}^{N-1}(x_\ell-x_m)q_\ell\wedge q_m\right),$$

and hence mixes with $\mathcal{A}$ through the $(N-1)$ moduli-parameter integrations.

The loop momentum and overall moduli integrals can be calculated explicitly. The loop momentum integral yields

$$\int\frac{\mathrm{d}^dk}{(2\pi)^d}\exp\left(-\alpha k^2+\mathrm{i}q_N\wedge k\right) = \left(\frac{1}{4\pi\alpha}\right)^{d/2}\exp\left(-\frac{q_N\circ q_N}{4\alpha}\right), \tag{28}$$

while the α-moduli integral yields

$$\int_0^\infty \mathrm{d}\alpha\alpha^{N-1-(d/2)}\exp\left[-m^2\alpha-\frac{q_N\circ q_N}{4\alpha}\right] = 2^{(d/2)-N+1}\left(\frac{q_N\circ q_N}{m^2}\right)^{(N/2)-(d/4)} \times\boldsymbol{K}_{(d/2)-N}\left(m|q_N\circ q_N|^{1/2}\right). \tag{29}$$

The remaining integrals over $(N-1)$ moduli parameters are given by

$$\widetilde{\mathcal{B}}^{(\mathrm{N})}_{\mathrm{total}} = \exp\left(\frac{\mathrm{i}}{2}\sum_{\ell<m}^{N-1}q_\ell\wedge q_m\right)\times\int_0^1\mathrm{d}x_1\int_0^{x_1}\mathrm{d}x_2\cdots\int_0^{x_{\mathrm{N}-2}}\mathrm{d}x_{N-1}\times\exp\left(-\mathrm{i}\sum_{\ell<m}^{N-1}(x_\ell-x_m)q_\ell\wedge q_m\right)+(\text{permutations}).$$

The (permutations) part refers to $(N-1)!$ diagrams of the same sort, differing from one another by permutation of the $(N-1)$ untwisted interaction vertices. Half of these diagrams are Hermitian conjugates of the other, corresponding to topological reversal between the inner and the outer sides. The complete non-planar N-point correlation function with a single twisted vertex is given by all possible cyclic permutations of the twisted interaction vertex with the $(N-1)$ untwisted ones.

For $N = 3$ correlation function, two diagrams of ordering (1-2-[3]) and (2-1-[3]), in which [3] refers to the twisted vertex, contribute. The result is

$$\widetilde{\mathcal{B}}^{(3)}_{\mathrm{total}} = \mathrm{e}^{(\mathrm{i}/2)q_1\wedge q_2}\int_0^1\mathrm{d}x\int_0^x\mathrm{d}y\exp(-\mathrm{i}(x-y)q_1\wedge q_2) + (1\longleftrightarrow 2) = \sin\left(\frac{q_1\wedge q_2}{2}\right)\Big/\left(\frac{q_1\wedge q_2}{2}\right), \tag{30}$$

yielding precisely the generalized star product $\star_2$. Note that, in deriving the result, higher powers of the momenta cancel [21], leaving only the $\star_2$ part.

For $N = 4$ correlation function, there are 3! diagrams. Three diagrams with ordering (1-2-3-[4]), (1-2-[4]-3) and (1-[4]-2-3) contribute, while the other three diagrams (1-[4]-3-2), (1-3-[4]-2) and (1-3-2-[4]) are Hermitian conjugates, respectively. Summing them up, one finds

$$\begin{aligned}
&\mathrm{e}^{(\mathrm{i}/2)(q_1\wedge q_2+q_1\wedge q_3+q_2\wedge q_3)}\int\cdots\int \\
&\times\mathrm{e}^{-\mathrm{i}[(x_1-x_2)q_1\wedge q_2+(x_1-x_3)q_1\wedge q_3+(x_2-x_3)q_2\wedge q_3]} \\
&+\mathrm{e}^{(\mathrm{i}/2)(q_3\wedge q_1+q_3\wedge q_1+q_1\wedge q_2)}\int\cdots\int \\
&\times\mathrm{e}^{-\mathrm{i}[(x_1-x_2)q_3\wedge q_1+(x_1-x_3)q_3\wedge q_2+(x_2-x_3)q_1\wedge q_2]} \\
&+\mathrm{e}^{(\mathrm{i}/2)(q_2\wedge q_3+q_2\wedge q_1+q_3\wedge q_1)}\int\cdots\int \\
&\times\mathrm{e}^{-\mathrm{i}[(x_1-x_2)q_2\wedge q_3+(x_1-x_3)q_2\wedge q_1+(x_2-x_3)q_3\wedge q_1]} \\
&+(\text{h.c.}) \\
&=\left(\sin\frac{q_2\wedge q_3}{2}\Big/\frac{(q_1+q_2)\wedge q_3}{2}\right) \\
&\times\left(\sin\frac{q_1\wedge(q_2+q_3)}{2}\Big/\frac{q_1\wedge(q_2+q_3)}{2}\right)+(1\longleftrightarrow 2),
\end{aligned}$$

yielding precisely the generalized $\star_3$ product.

The $N = 4$ correlation function contains another type of non-planar diagram, in which two interaction vertices are twisted. It consists of six diagrams of distinct permutations, among which half are Hermitian conjugates of the others. Adding them, one easily finds

$$\widetilde{\mathcal{B}}^{(4)}_{\text{total}} = \left(-\frac{\lambda}{2}\right)^4 (2\pi)^d \widetilde{\mathcal{K}}_{4-(d/2)}((q_3+q_4)\circ(q_3+q_4)) \times \left(\frac{\sin\frac{q_1\wedge q_2}{2}}{\frac{q_1\wedge q_2}{2}}\right)\left(\frac{\sin\frac{q_3\wedge q_4}{2}}{\frac{q_3\wedge q_4}{2}}\right),$$

viz. a product of two $\star_2$'s.

Hence, up to $\mathcal{O}(\lambda^4)$, after Wick rotation back to Minkowski spacetime, the non-planar part of the one-loop effective action is given by

$$\begin{aligned}\Gamma_{\text{np}}[\Phi] &= \frac{1}{3!}\left(-\frac{\lambda}{4}\right)^3 \int \mathrm{d}^d x \\ &\times \left(3[\Phi(x)\Phi(x)]_{\star_2}\mathcal{K}_{3-(d/2)}\left(-\partial_x\circ\partial_x\right)\Phi(x)\right) \\ &+\frac{1}{4!}\left(-\frac{\lambda}{4}\right)^4 \int \mathrm{d}^d x\, (3[\Phi(x)\Phi(x)]_{\star_2} \\ &\times\mathcal{K}_{4-(d/2)}(-(\partial_{x_1}+\partial_{x_2})\circ(\partial_{x_1}+\partial_{x_2})) \\ &\times[\Phi(x_1)\Phi(x_2)]_{\star_2} \\ &+4\left[\Phi(x)\Phi(x)\Phi(x)\right]_{\star_3}\mathcal{K}_{4-(d/2)}\left(-\partial_x\circ\partial_x\right)\Phi(x)\right) \\ &+\cdots,\end{aligned} \tag{31}$$

where, for the second line, $x_1, x_2 \to x$ is assumed in the end. The final expression is in complete agreement with the power-series expansion of (7), as given in (26).

3.3 Exact result: Saddle-point method

To substantiate the computations in the previous two subsections, we will now provide an intuitively transparent procedure for evaluating the Feynman diagrammatics exactly. Extending the calculation in the previous subsection, the N-point, one-particle-irreducible Green function, (17), is computed and found to be

$$\Gamma_N[\Phi] = \hbar\left(\frac{|\theta\cdot\boldsymbol{k}|}{2}\right)^N \sum_{N_1+N_2=N}\frac{C_{\{N\}}}{(4\pi)^{d/2}}\int_0^\infty \frac{\mathrm{d}T}{T}T^{-(d/2)+N} \times \exp\left[-m^2T - \frac{\boldsymbol{k}\circ\boldsymbol{k}}{4T}\right] J_{N_1}(\theta\cdot\boldsymbol{k})J_{N_2}(-\theta\cdot\boldsymbol{k}).$$

Here, we have denoted the N-dependent combinatoric factor $C_{\{N\}}$, and we divided N external momenta into two groups: $\{p_1,\cdots p_{N_1}\}$ and $\{q_1,\cdots,q_{N_2}\}$ and defined $k = \sum_{i=1}^{N_1} p_i = -\sum_{i=1}^{N_2} q_i$. We have also introduced the kernel $J_{N_1}(\theta\cdot\boldsymbol{k})$ by

$$J_{N_1}(\theta\cdot\boldsymbol{k}) := \int_0^1\cdots\int_0^1 \mathrm{d}\tau_1\cdots\mathrm{d}\tau_{N_1} \times\exp\left[-\mathrm{i}\sum_{i=1}^{N_1}\tau_i\boldsymbol{p}_i\cdot|\theta\cdot\boldsymbol{k}| - \frac{\mathrm{i}}{2}\sum_{i<j=1}^{N_1}\varepsilon(\tau_{ij})\boldsymbol{p}_i\wedge\boldsymbol{p}_j\right], \tag{32}$$

in which $\tau_i \sim \tau_i + 1$, $(i = 1, 2, \cdots)$ denote the moduli parameter for the ith marked point around the one-loop vacuum diagram in Fig. 1, where a background field Φ with momentum $\boldsymbol{p}_i$ is inserted, and $\tau_{ij} := (\tau_i - \tau_j)$. Similarly, $J_{N_2}(-|\theta\cdot\boldsymbol{k}|, \{\boldsymbol{q}_j\})$ is defined around the other boundary of the vacuum diagram. One can check straightforwardly that (32) is nothing but the momentum-space representation of the $\star_N$-product[4].

Remarkably, in the limit of (20) and (21), the T-moduli integral can be evaluated via the saddle-point method, whose result is precisely the same as (26) and (31)! First, the saddle point is located at $T = |\theta\cdot\boldsymbol{k}|/2m$. Moreover, the leading-order correction is from the Gaussian integral over the quadratic variance in the exponent, and is given by $(\pi|\theta\cdot\boldsymbol{k}|/m^3)^{1/2}$. Thus, putting all these pieces together, the one-loop, N-point Green function is obtained:

$$\begin{aligned}\Gamma_N[\Phi] &= \hbar(4\pi)^{-d/2}\left(\frac{|\theta\cdot\boldsymbol{k}|}{2m}\right)^{-d/2}\left(\frac{2\pi}{m|\theta\cdot\boldsymbol{k}|}\right)^{1/2}\mathrm{e}^{-m|\theta\cdot\boldsymbol{k}|} \\ &\times\sum_{N_1+N_2=N} C_{\{N\}}\left\{(-g)^{N_1}|\theta\cdot\boldsymbol{k}|^{N_1}J_{N_1}(\theta\cdot\boldsymbol{k})\right\} \\ &\times\left\{(-g)^{N_2}|\theta\cdot\boldsymbol{k}|^{N_2}J_{N_2}(-\theta\cdot\boldsymbol{k})\right\}.\end{aligned} \tag{33}$$

The factorized expression permits resummation of the double sum over N_1 and N_2. Indeed, taking carefully into account of the combinatorial factor, $C_{\{N\}} = (1/2)(N!/N_1!N_2!)$, we find that each factor involving the $\star_N$-kernel J_N's are exponentiated into the *scalar* open Wilson lines. Convoluting the Green functions with the background Φ fields and summing over N, the result is

$$\Gamma_{1-\text{loop}}[W(\Phi)] = \frac{\hbar}{2}\int\frac{\mathrm{d}^2\boldsymbol{k}_1}{(2\pi)^2}\frac{\mathrm{d}^2\boldsymbol{k}_2}{(2\pi)^2}(2\pi)^2\delta^2(\boldsymbol{k}_1+\boldsymbol{k}_2) \times W_{\boldsymbol{k}_1}[\Phi]\widetilde{\mathcal{K}_{-d/2}}(-\boldsymbol{k}_1\circ\boldsymbol{k}_2)W_{\boldsymbol{k}_2}[\Phi],$$

yielding precisely the proclaimed result (7). Moreover, we also obtain the series expansion for the scalar open Wilson line on a *straight* contour:

$$\begin{aligned}W_k[\Phi] &= \sum_{N=0}^{\infty}\frac{1}{N!}(-g|\theta\cdot\boldsymbol{k}|)^N \\ &\times\int\frac{\mathrm{d}^2\boldsymbol{p}_1}{(2\pi)^2}\cdots\int\frac{\mathrm{d}^2\boldsymbol{p}_N}{(2\pi)^2}(2\pi)^2\delta^2\left(\boldsymbol{p}_1+\cdots+\boldsymbol{p}_N-\boldsymbol{k}\right) \\ &\times[J_N\left(\theta\cdot\boldsymbol{k},\{\boldsymbol{p}_i\}\right)\cdot\widetilde{\Phi}(\boldsymbol{p}_1)\cdots\widetilde{\Phi}(\boldsymbol{p}_N)],\end{aligned} \tag{34}$$

viz. an expression as a convolution of the $\star_N$-product kernel J_N, (32) with background scalar field.

We are confident that the foregoing saddle-point analysis is the most conclusive proof to the exponentiation of the non-planar effective action, expressed in terms of the scalar open Wilson lines, and hence constitutes the *central result* of this work.

[4] As $\boldsymbol{k}\cdot(\theta\cdot\boldsymbol{k}) = 0$, the kernel is invariant under a uniform shift of all moduli parameters: $\tau_i \to \tau_i +$ (constant). Using the invariance, one can shift the τ-integral domain in (32) to $[-1/2, +1/2]$. This choice is preferred, as it displays the Hermiticity property of J_N manifestly

3.4 Further remarks

We close this section with two remarks concerning our result. First, the contribution of the planar part to the one-loop effective action can be deduced straightforwardly by replacing, in (7) and (26), $|\theta \cdot \boldsymbol{k}|$ into $2/\Lambda_{\rm UV}$, where $\Lambda_{\rm UV}$ is the ultraviolet cutoff, and all the generalized $\star_n$ products into Moyal's $\star$-product. In the limit $\Lambda_{\rm UV} \to \infty$, utilizing the Taylor expansion of the modified Bessel function $\boldsymbol{K}_n$, one easily recognizes that the planar part of the effective action reduces to a "Coleman–Weinberg"-type potential and appears not to be expressible in terms of the open Wilson line operators, even including those of *zero* momentum.

Second, from the explicit Feynman diagrammatics, as in (31), zero-, and one-point Green functions are trivially planar. On the other hand, the proclaimed result (7) includes them explicitly. How can this be true? The point is that these lower-point Green functions, as computed by the standard Feynman diagrammatics, are UV divergent, whose maximal divergence is dictated by power-counting. The lower-point, non-planar Green functions, as extracted from (18), are IR divergent. If we compare the degree of the divergence, we find that the IR divergences in the non-planar expressions are always lower than the UV divergences in the planar ones. Thus, what (7) has prescribed implicitly is that part of the sub-leading UV divergences in the planar parts are reinterpreted as IR divergences. This also implies that, at least for the lower-point functions, the renormalization prescription can be different from those of the commutative field theories.

Third, one might consider the results in this work trifling as, in the low-energy, large non-commutativity limit, (20) and (21), the kernel $\widetilde{\mathcal{K}}_{-d/2} \sim \exp(-m|\theta \cdot \boldsymbol{k}|)$ is exponentially suppressed. Quite to the contrary, we actually believe that the exponential suppression indicates a sort of *holography* in terms of a gravity theory. Indeed, in the context of non-commutative D3-branes, as shown in [4], the Green's functions of supergravity fields exhibit precisely the same, exponentially suppressed propagation in the region where the non-commutativity is important, viz. the dipole-dominated, "UV–IR proportionality" region. We view this as evidence that, in non-commutative field theories, the dipoles are provided by open Wilson line operators and that their dynamics describes, via holography, gravitational interactions. The latter then defines *an effective field theory of the dipoles.*

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